No. III.

To determine the true Place of a Planet, in an Elliptical Orbit, directly from the mean Anomaly, by Converging Series, by DAVID RITTENHOUSE, L. L. D. *President* A. P. S.

Read Feb. 5, 1796.

LET $x =$ the eccentricity, y the mean anomaly in the arch of a circle the radius whereof is 1. And a, an arch required.

For the upper half of the orbit, let $\frac{x}{x+1}$, $= n$, and $\frac{y}{x+1}$, $= z$.

Then $a = z + \frac{n}{6}z^3 + \overline{\frac{nn}{12} - \frac{n}{120}}\,z^5 + \overline{\frac{nnn}{18} - \frac{nn}{90} + \frac{n}{5040}}\,z^7 + \overline{\frac{55n^4}{1296} - \frac{11n^3}{864} + \frac{41nn}{60480} - \frac{n}{362880}}\,z^9$ &c.

Find the log. of the natural cosine of a, and the log. of the same cosine $+ x$, and add the difference of these two logarithms, and likewise the complement of the log. of the conj. semidiameter, and the log. cotang. of a, together, the sum will be the log. cotang. of the true anomaly.

For the lower half of the Orbit.

Let y, be the mean anomaly from the lower apsis, $\frac{x}{1-x} = n$, and $\frac{y}{1-x} = z$.

Then $a = z - \frac{n}{6}z^3 + \overline{\frac{nn}{12} + \frac{n}{120}}\,z^5 - \overline{\frac{nnn}{18} + \frac{nn}{90} + \frac{n}{5040}}\,z^7 + \overline{\frac{55n^4}{1296} + \frac{11n^3}{864} + \frac{41nn}{60480} + \frac{n}{362880}}\,z^9$ &c.

Take the difference between the log. of the nat. cosine of a, and the log. of the same cosine $- x$, and subtract this diff. — the comp. above mentioned from the log. cotang. of a, the remainder is the log. cotang. of the true anomaly, counted from the lower apsis.

If the co-efficients prefixed to the powers of z, be computed for any particular orbit, and their logarithms used instead of the numbers themselves, the calculation will afterwards be very simple for any degree of mean anomaly in that orbit, as will appear by the following example.

In the very elaborate tables of Mr. Zach, published in 1792 the eccentricity of the Earth's orbit is assumed .0167923, consequently log. of the lesser semidiameter will be — 1.9999387, its

complement

complement .0000613, and the log. of $n = -2.2178779$ and the ſeries for the upper half of the orbit will be,

$$a = a + -3.4397266.\,z^3$$
$$- -4.0603053.\,z^5$$
$$+ -7.6959472.\,z^7$$
$$- -8.9982252.\,z^9 \text{ \&c.}$$

The negative ſign prefixed to theſe logarithms affects the index only.

For the lower half, $a = z - -3.4543136.\,z^3$

$$+ -4.2217638.\,z^5$$
$$- -6.8392607.\,z^7$$
$$+ -7.4939405.\,z^9 \text{ \&c.}$$

For the logarithm of the diſtance in any part of the Orbit.

To the log. ſine of a, add the comp. of the log. ſine of the true anomaly — the comp. of the log. of the conjugate ſemidiam. the ſum will be the true log. of the diſtance.

Example of the Calculation.

The Sun's mean anomaly being II^s 6° 30′ required the true anomaly or equation.

Arch of 66° 30′ = 1.16064395 =

$$y,\ \text{log.} = .0646990$$
$$\text{Sub. } 1 + x \text{ log. } \underline{.0072323}$$
$$\log z = \underline{.0574667}\ z = 1.1414758$$
$$z^3 = .1724001$$
$$+ -\underline{3.4397266}$$
$$-3.6121267 = +.0040938.01$$

$$z^5 = .2873335$$
$$+ -\underline{4.0603053}$$
$$-4.3476388 = -.0002226.58$$

$$z^7 = .4022669$$
$$+ -\underline{7.6959472}$$
$$-6.0982141 = +.0000012.54$$

$$z^9 = .5172003$$
$$+ -\underline{8.9982252}$$
$$-7.5154255 = -\ \underline{.0000003.27}$$
$$+\ 1.1455708.55$$
$$-\ \underline{.0002229.85}$$
$$a = 1.1453478.7 = 65°\,37'\,24''.96$$

Log.

Nat. cosine of $a = .4127292.5$ log. $= -1.6156652$
Eccent. $+\ .0167923.$

$.4295215.5$ log. $= -1.6329849$

Diff. log. $= .0173197$
Comp. of conj. semidiam. log. $= .\quad 613$
Cotang. of a, $65^\circ\ 37'\ 24''.96 = 9.6562166$

Cotang. of true anomaly $64^\circ\ 45'\ 0''.8 = 9.6735976$

Log. sine $a = 9.9594487$
Log. sine $64^\circ\ 45'\ 0''.8$ comp $= .0436122$

10.0030609
Comp. of log. conj. — 613

Log. distance $= 10.0029996$

Hence the equation is $1^\circ\ 44'\ 59''.2$. In Zach's tables it is $1^\circ\ 44'\ 59''.25$.

The series above given converge slowly when the mean anomaly is near 3 S. or 9 S. In this case the true anomaly may be obtained with great accuracy by a series derived from that which expresses the cosine in terms of its correspondent arch, as follows,

Subtract the eccentricity from the mean anomaly and call the remainder R. Let the difference between R and 90° be $= z$,

Then $R + \frac{x}{2} zz \pm \frac{xx}{2} z^3 - \overline{\frac{x}{24} - \frac{5xxx}{8}}\, z^4 \mp \overline{\frac{xx}{8} - \frac{7x^4}{8}}\, z^5 + \overline{\frac{x}{720} - \frac{7xxx}{24} + \frac{21x^5}{16}}\, z^6$ &c. $= a$

And a will be the true anom. counted from the upper apsis.

For the Earth's Orbit.

$$
\begin{aligned}
a = R & + -3.9240802.\ zz \\
& \pm -4.1491904.\ z^3 \\
& - -4.8430581.\ z^4 \\
& \mp -5.5462722.\ z^5 \\
& + -5.3413007.\ z^6 \\
& \pm -6.4890406.\ z^7
\end{aligned}
$$

The uppermost of the signs prefixed to the 3d 5th and 7th powers of z must be used when R exceeds 90° and the lower sign when R is less than 90°.

Examples of Calculation.

Sun's mean anomaly 60° 50′ required the true anom. and equation.

$$
\begin{array}{rl}
60^\circ\ 50' = & 1.06174196 \\
-x = & .0167923 \\ \hline
R = & 1.04494966 \\
90^\circ = & 1.57079633 \\ \hline
z = & .52584667 \\ \hline
\text{Log. } z = & -1.7208591 \\
zz = & -1.4417182 \\
+ & -3.9240802 \\ \hline
& -3.3657984\ + \\ \hline
z^3 = & -1.1625773 \\
+ & -4.1491904 \\ \hline
& -5.3117677\ - \\ \hline
z^4 = & -2.8834364 \\
+ & -4.8430581 \\ \hline
& -5.7264945\ - \\ \hline\hline
\end{array}
\qquad
\begin{array}{rl}
z^5 = & -2.6042945 \\
+ & -5.5462722 \\ \hline
& -6.1505667\ + \\ \hline
z^6 = & -2.3251546 \\
+ & -5.3413007 \\ \hline
& -7.6664553\ + \\ \hline
z^7 = & -2.0460137 \\
+ & -6.4890406 \\ \hline
& -8.5350543\ - \\
\end{array}
$$

$$
\begin{array}{rl}
R = & 1.04494966 \\
+ & 232166 \\
+ & 141 \\
+ & 46 \\ \hline
& 1.04727319 \\
\text{Sum of} - & 7380 \\ \hline
a = & 1.04719939 = 60^\circ\ 0'\ 0''.38 \\
\end{array}
$$

$$
\begin{array}{rrl}
\text{Nat. cofine} = .4999984 & \text{Log.} = & .6989686 \\
+x\ .0167923 & & \\ \hline
.5167907 & \text{Log.} = & .7133147 \\ \hline
& \text{Diff.} = & .0143461 \\
& + & 613 \\
\text{Cotang. } 60^\circ\ 0'\ 0''.38 & = & 9.7614376 \\ \hline
\text{Cotang. true anomaly } 59^\circ\ 10'\ 13''.1 & = & 9.7758450 \\
\end{array}
$$

Equation — 1° 39′ 46″.9.

The

The Sun's mean anomaly 120° 50′ required the true anomaly and equation.

$$
\begin{aligned}
120^\circ\ 50' &= 2.10893951\\
-x &= \underline{\ .0167923}\\
R &= 2.09214721\\
90^\circ &= \underline{1.57079633}\\
z &= \ .52135088
\end{aligned}
$$

$$
\begin{aligned}
\text{Log. } z &= -\underline{1.7171301}\\
zz &= -1.4342602\\
&+-\underline{3.9240802}\\
&\quad -3.3583404 = +.00228213\\[1ex]
z^3 &= -1.1513903\\
&+-\underline{4.1491904}\\
&\quad -5.3005807 = +.00001998\\[1ex]
z^4 &= -2.8685204\\
&+-\underline{4.8430581}\\
&\quad -5.7115785 = -.00005147\\[1ex]
z^5 &= -2.5856505\\
&+-\underline{5.5462722}\\
&\quad -6.1319227 = -.00000135\\[1ex]
z^6 &= -2.3027806\\
&+-\underline{5.3413007}\\
&\quad -7.6440813 = +\underline{.00000044}\\
&\qquad \text{Sum of } + = .00230255\\
&\qquad \text{Sum of } - = -\underline{\qquad 5282}\\
&\qquad\qquad +.00224973
\end{aligned}
$$

$$
\begin{aligned}
R &= 2.09214721\\
+ &\quad \underline{\quad 224973}\\
&\ 2.09439694 = 120^\circ\ 0'\ 0''.38
\end{aligned}
$$

Comp.

$$
\begin{array}{rr}
 & \text{Log.} \\
\text{Comp. to } 180^\circ = a = 59^\circ\ 59'\ 59''.62 \text{ cos. } .5000015.9 = & .6989714 \\
-x = \underline{.0167923.} & \\
.4832092.9 = & \underline{.6841352} \\
\text{diff.} = & \underline{.0148362} \\
\text{Cotang. } 59^\circ\ 59'\ 59''.52 = & \underline{9.7614412} \\
 & 9.7466050 \\
+ & \underline{613} \\
\text{Tang. } 29^\circ\ 9'\ 48'' = & 9.7466663 \\
+\ \underline{90} & \\
\text{True anomaly } 119^\circ\ 9'\ 48'' & \\
\text{Mean anom. } \underline{120\ 50\ 00} & \\
\text{Equation} - 1^\circ\ 40'\ 12'' &
\end{array}
$$

If the 1ſt and 6th 60° of mean anomaly in the Earth's orbit be computed by the firſt ſeries, the 3d and 4th 60° by the ſecond ſeries, and the 2d and 5th by the laſt ſeries, no more than the firſt 3 terms containing powers of z, need be uſed, for the equation cannot be had true to $\frac{1}{100}$ of a ſecond without tables of logarithms carried farther than to 7 places.

N°. IV.

On the Improvement of Time-keepers, by DAVID RITTENHOUSE, L. L. D. *Preſident of the Society.*

Read Nov. 7, 1794.

THE invention and conſtruction of time-keepers may be reckoned amongſt the moſt ſucceſsful exertions of human genius. Pendulum clocks eſpecially, have been made to meaſure time with aſtoniſhing accuracy; and if there are ſtill ſome cauſes of inequality in their motions, the united efforts of mechaniſm, philoſophy and mathematics will probably in time remove them.

The laſt and leaſt of thoſe cauſes, which perhaps may be worthy of notice when all others of more importance are